Pricing swaps and options on quadratic variation under stochastic time change models

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- When replicating variance swaps (a log contract) at least three sources of errors could occur in practice:
  1. The analytical error due to jumps in the asset price.
  2. Interpolation/extrapolation error from the finite option quotes available to the continuum of options needed in the replication.
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  2. Interpolation/extrapolation error from the finite option quotes available to the continuum of options needed in the replication.
  3. Errors in computing the realized return variance.
- Going from hedging to modeling, we come up to a known observation that simple models are not able to replicate the price of the quadratic variation contract for all maturities.

The idea of the work (continue)

- Therefore, one can see a steadfast interest to applying more sophisticated jump-diffusion and stochastic volatility models to pricing swaps and options on the quadratic variation. Among multiple papers on the subject, note the following: Schoutens(2005), Carr & Lee (2003), Carr, Geman, Madan, Yor (2005), Gatheral & Friz (2005).

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- Analytical and semi-analytical (like FFT) results are available only for simplest models. For instance, Swichchuk (2004) uses the change-of-time method for the Heston model to derive explicit formulas for variance and volatility swaps for financial markets with stochastic volatility following the CIR process. Also Carr et all (2005) proposed a method of pricing options on quadratic variation via the Laplace transform, but this methods has some serious pitfalls.

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- A similar class of models uses stochastic time change and, thus, operates with time-changed Levy processes. Usually, Monte-Carlo methods are used to price the quadratic variation products within these models.
- In the present paper we consider a class of models that are known to be able to capture at least the average behavior of the implied volatilities of the stock price across moneyness and maturity - time-changed Levy processes. We derive an analytical expression for the fair value of the quadratic variation and volatility swap contracts as well as use the approach similar to that of Carr & Madan (1999) to price options on these products.

Quadratic variation and forward characteristic function

Quadratic variation

- Quadratic variation of the stochastic process s_t is defined as follows

$$QV(s_t) = \mathbb{E}_{\mathbb{Q}} \left[\sum_{i=1}^{N} \left[\log \left(\frac{s_{t_i}}{s_{t_{i-1}}} \right) \right]^2 \right] \quad (1)$$

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- In case of N discrete observations over the life of the contract with maturity T, annualized quadratic variation of the stochastic process s_t is then

$$\overline{QV}(s_t) = \mathbb{E}_{\mathbb{Q}}\left[\frac{k}{N}\sum_{i=1}^{N}\left(\log\frac{s_{t_i}}{s_{t_{i-1}}}\right)^2\right] = \frac{k}{N}QV(s_t) \tag{2}$$

where k is the number of observations per year.

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where k is the number of observations per year.

- Suppose the observations are uniformly distributed over $(0, T)$ with $\tau = t_i - t_{i-1} = const, \forall i = 1, N$. Then

$$\overline{QV}(s_t) \equiv \frac{1}{T}\sum_{i=1}^{N}\mathbb{E}_{\mathbb{Q}}\left[(s_{t_i} - s_{t_{i-1}})^2\right] \quad (3)$$

Forward characteristic function

- Quadratic variation is often used as a measure of realized variance. Moreover, modern variance and volatility swap contracts in fact are written as a contract on the quadratic variation because i) this is a quantity that is really observed at the market, and ii) for models with no jumps the quadratic variance exactly coincides with the realized variance.

Forward characteristic function

- As shown by Hong (2004) an alternative representation of the quadratic variation could be obtained via a forward characteristic function. The idea is as follows.
  Let us define a forward characteristic function

$$\phi_{t,T} \equiv \mathbb{E}_{\mathbb{Q}} \left[\exp(ius_{t,T}) | s_0, \nu_0 \right] \equiv \int_{-\infty}^{\infty} e^{ius} q_{t,T}(s) ds, \tag{4}$$

  where $s_{t,T} = s_T - s_t$ and $q_{t,T}$ is the $\mathbb{Q}$-density of $s_{t,T}$ conditional on the initial time state

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- From Eq. (3) and Eq. (4) we obtain

$$\overline{QV}(s_t) \equiv \frac{1}{T} \sum_{i=1}^{N} \mathbb{E}_{\mathbb{Q}} \left[(s_{t_i} - s_{t_{i-1}})^2 \right] = \frac{1}{T} \sum_{i=1}^{N} \mathbb{E}_{\mathbb{Q}} \left[s^2_{t_i, t_{i-1}} \right] = -\frac{1}{T} \sum_{i=1}^{N} \frac{\partial^2 \phi_{t_i, t_{i-1}}(u)}{\partial u^2} \bigg|_{u=0}. \tag{6}$$

Analytical expression for the forward characteristic function

Forward characteristic function

- Let us first remind that a general Lévy process X_T has its characteristic function represented in the form

$$\phi_X(u) = \mathbb{E}_{\mathbb{Q}}\left[e^{iuX_T}\right] = e^{-T \quad x(u)}, \tag{7}$$

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- For time-changed Lévy process, Carr and Wu (2004) show that the generalized Fourier transform can be converted into the Laplace transform of the time change under a new, complex-valued measure, i.e. the time-changed process $Y_t = X_{\mathbf{T}_t}$ has the characteristic function

$$\phi_{Y_t}(u) = \mathbb{E}_{\mathbb{Q}}\left[e^{iuX_{\mathbf{T}_t}}\right] = \mathbb{E}_{\mathbb{M}}\left[e^{-\mathbf{T}_t \quad _x(u)}\right] = \mathcal{L}^u_{\mathbf{T}_t}\left(\quad _x(u)\right), \tag{8}$$

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where the expectation and the Laplace transform are computed under a new complex-valued measure $\mathbb{M}$.

- The measure $\mathbb{M}$ is absolutely continuous with respect to the risk-neutral measure $\mathbb{Q}$ and is defined by a complex-valued exponential martingale

$$\mathbb{D}_T(u) \equiv \left.\frac{d\mathbb{M}}{d\mathbb{Q}}\right|_T = \exp\left[iuY_T + \mathbf{T}_T \ _x(u)\right], \tag{9}$$

where $\mathbb{D}_T$ is the Radon-Nikodym derivative of the new measure with respect to the risk neutral measure up to time horizon T.

FCF (continue)

- Further we again follow to the idea of Hong (2004). For the process Eq. (7) we need to obtain the forward characteristic function which is

$$\phi_{t,T}(u) \equiv \mathbb{E}_{\mathbb{Q}}\left[e^{iu(s_T - s_t)} \middle| \mathcal{F}_0 \right] = e^{iu(r-q)(T-t)} \mathbb{E}_{\mathbb{Q}}\left[e^{iu(\mathbf{Y}_T - \mathbf{Y}_t)} \middle| \mathcal{F}_0 \right], \tag{10}$$

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- From the Eq. (10) one has

$$\begin{aligned} \mathbb{E}_{\mathbb{Q}}\left[e^{iu(\mathbf{Y}_T-\mathbf{Y}_t)}\middle|\mathcal{F}_0\right] &= \mathbb{E}_{\mathbb{Q}}\left[\mathbb{E}_{\mathbb{Q}}\left[e^{iu(\mathbf{Y}_T-\mathbf{Y}_t)}\middle|\mathcal{F}_t\right]\right] \\ &= \mathbb{E}_{\mathbb{Q}}\left[\mathbb{E}_{\mathbb{Q}}\left[e^{iu(\mathbf{Y}_T-\mathbf{Y}_t)+(\mathbf{T}_T-\mathbf{T}_t)\ x(u)-(\mathbf{T}_T-\mathbf{T}_t)\ x(u)}\middle|\mathcal{F}_t\right]\right] \\ &= \left[\mathbb{E}_{\mathbb{Q}}\left[\frac{\mathbb{M}_T}{\mathbb{M}_t}e^{-(\mathbf{T}_T-\mathbf{T}_t)\ x(u)}\middle|\mathcal{F}_t\right]\right] = \mathbb{E}_{\mathbb{Q}}\left[\mathbb{E}_{\mathbb{M}}\left[e^{-(\mathbf{T}_T-\mathbf{T}_t)\ x(u)}\middle|\mathcal{F}_t\right]\right] \end{aligned} \tag{11}$$

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  For Markovian arrival rates ν the inner expectation will be a function of $\nu(t)$ only.

- Now let us consider a time-homogeneous time-change processes, for instance, CIR process with constant coefficients. With the allowance of the Eq. (8) the last expression could be rewritten as

$$\mathbb{E}_{\mathbb{Q}}\left[\mathbb{E}_{\mathbb{M}}\left[e^{-(\mathbf{T}_T - \mathbf{T}_t)\ x(u)} \middle| \mathcal{F}_t \right]\right] = \mathbb{E}_{\mathbb{Q}}\left[\mathbb{E}_{\mathbb{M}}\left[e^{-\ x(u)\int_t^T \nu(s)ds} \middle| \nu_t \right]\right] = \mathbb{E}_{\mathbb{Q}}\left[\mathcal{L}_\tau^u(\ x(u)) \middle| \nu_0 \right], \tag{12}$$

  where $\tau \equiv T - t$.

Affine arrival rates

Affine rates

- Now for all the arrival rates that are affine, the Laplace transform $\mathcal{L}^u_\tau(\quad_x(u))$ is also an exponential affine function in ν_t

$$\mathcal{L}^u_\tau(\quad_x(u)) = \exp\left[\alpha(\tau, \quad_x(u)) + \beta(\tau, \quad_x(u))\nu_t\right], \tag{13}$$

and hence

$$\begin{aligned}\phi_{t,T}(u) &= e^{iu(r-q)\tau}\mathbb{E}_{\mathbb{Q}}\left[e^{iu(\mathbf{Y}_T - \mathbf{Y}_t)}\middle|\mathcal{F}_0\right] = e^{iu(r-q)\tau}\mathbb{E}_{\mathbb{Q}}\left[\exp\left[\alpha(\tau, \quad_x(u)) + \beta(\tau, \quad_x(u))\nu_t\right]\middle|\nu_0\right] \\ &= e^{iu(r-q)\tau}e^{\quad(\tau, \quad_x(u))}\mathbb{E}_{\mathbb{Q}}\left[e^{\quad(\tau, \quad_x(u))\nu_t}\middle|\nu_0\right] = e^{iu(r-q)\tau}e^{\quad(\tau, \quad_x(u))}\phi_{\nu_t}\left(-i\beta(\tau, \quad_x(u))\nu_t\right).\end{aligned} \tag{14}$$

Here as $\phi_{\nu_t}()$ we denote the generalized characteristic function of the activity rate process under the risk neutral measure $\mathbb{Q}$.

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- **Example: CIR clock change.** In the case of the CIR clock change the conditional Laplace transform (or moment generation function) of the CIR process

$$\psi_{t,h}(v) = \mathbb{E}_\mathbb{Q}\left[e^{-vy_{t+h}}\middle|y_t\right], \quad v \geq 0 \tag{15}$$

can be found in a closed form (Heston). Since ν_t in our case is a positive process, the conditional Laplace transform characterizes the transition between dates t and $t+h$ (Feller 1971).

CIR clock change

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Now, expressions for $\alpha(\tau, \ _x(u))$ and $\beta(\tau, \ _x(u))$ in the case of the CIR time-change have been already found in Carr & Wu (2004) and read

$$\beta(\tau, \ _x(u)) = -\frac{2 \ _x(u)(1-e^{-\delta\tau})}{(\delta+\kappa^Q) + (\delta-\kappa^Q)e^{-\delta\tau}}, \qquad (18)$$

$$\alpha(\tau, \ _x(u)) = -\frac{\kappa^Q\theta}{\eta^2}\left[2\log\left(1 - \frac{\delta-\kappa^Q}{2\delta}(1-e^{-\delta\tau})\right) + (\delta-\kappa^Q)\tau\right],$$

where $\delta^2 = (\kappa^Q)^2 + 2 \ _x(u)\eta^2$, $\kappa^Q = \kappa - iu\eta\sigma\rho$ and σ is a constant volatility rate of the diffusion component of the process.

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- Further let us have a more close look at the Eq. (8). Suppose the distance between any two observations at time t_i and t_{i-1} is one day. Suppose also that these observations occur with no weekends and holidays. Then $\tau_i \equiv t_i - t_{i-1} = \tau = const$. Further we have to use the Eq. (17) with $t = t_{i-1}$ and $T = t_i$, substitute it into the Eq. (8), take second partial derivative and put $u = 0$.

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- Now we introduce an important observation that usually $\kappa\tau \ll 1$. Indeed, according to the results obtained for the Heston model calibrated to the market data the value of the mean-reversion coefficient κ lies in the range $0.01 - 30$. On the other hand, as it was already mentioned, we assume the distance between any two observations at time t_i and t_{i-1} to be one day, i.e $\tau = 1/365$. Therefore, the assumption $\kappa\tau \ll 1$ is provided with a high accuracy.

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- The above means that our problem of computing $\phi''_u(t_i, t_i+\tau)(u=0)$ has two small parameters - u and $\kappa\tau$. And, in principal, we could produce a double series expansion of $\phi''_u(t_i, t_i+\tau)$ on both these parameters. However, to make it more transparent, let us expand the Eq. (17) first into series on $\kappa\tau$ up to the linear terms (that can also be done with Mathematica).

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- Eventually for CIR clock change we arrive at the following result

$$-\frac{\partial^2 \phi_{t_i,t_{i-1}}(u)}{\partial u^2}\Big|_{u=0} = \frac{\partial^2 \quad x(u)}{\partial u^2}\Big|_{u=0} \left[\theta + (\nu_0-\theta)e^{-\kappa t_i}\right]\tau + O(\tau^2) \qquad (19)$$

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- Then from the Eq. (6) we obtain

$$\overline{QV}(s_t) = -\frac{1}{T}\sum_{i=1}^{N}\left(\frac{\partial^2 \phi_{t_i,t_{i-1}}(u)}{\partial u^2}\Big|_{u=0}\right) \approx \frac{1}{T}\int_0^T \left(\;_x)''_u(0)\left[\theta + (\nu_0-\theta)e^{-\kappa t}\right]\right) dt \tag{19}$$

$$= (\;_x)''_u(0)\left[\theta + (\nu_0-\theta)\frac{1-e^{-\kappa T}}{\kappa T}\right].$$

Some examples

CIR clock change. Examples

- **Heston model.** Let us remind that the familiar Heston model can be treated as the pure continuous Lévy component (pure lognormal diffusion process) with $\sigma = 1$ under the CIR time-changed clock. For the continuous diffusion process the characteristic exponent is (see, for instance, Carr & Wu (2004)) $\quad_x(u) = -i\mu u + \sigma^2 u^2/2$, therefore $(\quad_x)''_u(0) = 1$. Thus, we arrive at the well-known expression of the quadratic variation under the Heston model (see, for instance, Swishchuk (2004))

$$\overline{QV}(s_t) = \theta + (\nu_0 - \theta)\frac{1 - e^{-\kappa T}}{\kappa T} \tag{20}$$

CIR clock change. Examples

- **SSM model.** According to Carr & Wu (2004) let us consider a class of models that are known to be able to capture at least the average behavior of the implied volatilities of the stock price across moneyness and maturity. We use a complete stochastic basis defined on a risk-neutral probability measure $\mathbb{Q}$ under which the log return obeys a time-changed Lévy process

$$s_t \equiv \log S_t/S_0 = (r-q)t + \left(\mathbf{L}^R_{T^R_t} - \xi^R \mathbf{T}^R_t\right) + \left(\mathbf{L}^L_{T^L_t} - \xi^L \mathbf{T}^L_t\right), \tag{20}$$

where r, q denote continuously-compounded interest rate and dividend yield, both of which are assumed to be deterministic; $\mathbf{L}^R$ and $\mathbf{L}^L$ denote two Lévy processes that exhibit right (positive) and left (negative) skewness respectively; T^R_t and T^L_t denote two separate stochastic time changes applied to the Lévy components; ξ^R and ξ^L are known functions of the parameters governing these Lévy processes, chosen so that the exponentials of $\mathbf{L}^R_{T^R_t} - \xi^R \mathbf{T}^R_t$ and $\mathbf{L}^L_{T^L_t} - \xi^L \mathbf{T}^L_t$ are both $\mathbb{Q}$ martingales. Each Lévy component can has a diffusion component, and both must have a jump component to generate the required skewness.

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- First, by setting the unconditional weight of the two Lévy components equal to each other, we can obtain an unconditionally symmetric distribution with fat tails for the currency return under the risk-neutral measure. This unconditional property captures the relative symmetric feature of the sample averages of the implied volatility smile. Second, by applying separate time changes to the two components, aggregate return volatility can vary over time so that the model can generate stochastic volatility. Third, the relative weight of the two Lévy components can also vary over time due to the separate time change

SSM - continue

- For model design we make the following decomposition of the two Lévy components in the Eq. (20)

$$\mathbf{L}_t^R = J_t^R + \sigma^R W_t^R, \quad \mathbf{L}_t^L = J_t^L + \sigma^L W_t^L, \tag{21}$$

where (W_t^R, W_t^L) denote two independent standard Brownian motions and (J_t^R, J_t^L) denote two pure Lévy jump components with right and left skewness in distribution, respectively.

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- We assume a differentiable and therefore continuous time change and let

$$\nu_t^R \equiv \frac{\partial \mathbf{T}_t^R}{\partial t}, \quad \nu_t^L \equiv \frac{\partial \mathbf{T}_t^L}{\partial t}, \tag{22}$$

  denote the instantaneous activity rates of the two Lévy components. By definition $\mathbf{T}_t^R, \mathbf{T}_t^L$ have to be non-decreasing semi-martingales.

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- We model the two activity rates as a certain affine process. For instance, it could be a square-root processes of Heston (1993)

$$\begin{aligned} d\nu_t^R &= \kappa^R(\theta^R - \nu_t^R)dt + \eta^R\sqrt{\nu_t^R}dZ_t^R, \\ d\nu_t^L &= \kappa^L(\theta^L - \nu_t^L)dt + \eta^L\sqrt{\nu_t^L}dZ_t^L, \end{aligned} \tag{23}$$

  where in contrast to Carr & Wu (2004) we don't assume unconditional symmetry and therefore use different mean-reversion κ, long-run mean θ and volatility of volatility η parameters for left and right activity rates.

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- We allow the two Brownian motions (W_t^R, W_t^L) in the return process and the two Brownian motions (Z_t^R, Z_t^L) in the activity rates to be correlated as follows,

$$\rho^R dt = \mathbb{E}_{\mathbb{Q}}[dW_t^R dZ_t^R], \quad \rho^L dt = \mathbb{E}_{\mathbb{Q}}[dW_t^L dZ_t^L]. \tag{23}$$

  The four Brownian motions are assumed to be independent otherwise.

SSM (continue)

- Now assuming that the positive and negative jump components are driven by two different CIR stochastic clocks as in the Eq. (16), it could be shown in exactly same way as we did for the single time process, that the annualized fair strike $QV(s_t)(T)$ is now given by the expression

$$\overline{QV}(s_t)(T) = (\; {}^{L}_{x})''_{u}(0)\left[\theta^L + (\nu_0^L - \theta^L)\frac{1-e^{-\kappa^L T}}{\kappa^L T}\right] + (\; {}^{R}_{x})''_{u}(0)\left[\theta^R + (\nu_0^R - \theta^R)\frac{1-e^{-\kappa^R T}}{\kappa^R T}\right]. \tag{24}$$

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- So now we have two independent mean-reversion rates and two long-term run coefficients that can be used to provide a better fit for the long-term volatility level and the short-term volatility skew, similar to how this is done in the multifactor Heston (CIR) model.

Numerical experiments

Fair strike of the quadratic variation for three models

- **Heston model** An expression for the characteristic exponent of the Heston model reads

$$ {}_x(u) = -i\mu u + \frac{1}{2}\sigma^2 u^2, \qquad (25) $$

therefore ${}''_x(u)|_{u=0} = \sigma^2$.
The Heston model has 5 free parameters $\kappa, \theta, \eta, \rho, v_0$ that can be obtained by calibrating the model to European option prices. In doing so one can use an FFT method as in Carr and Madan (1999).

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- **SSM model.** To complete the description of the model we specify two jump components J_t^L and J_t^R using the following specification for the Lévy density (Carr & Wu (2004))

$$\mu^R(x) = \begin{cases} \lambda^R e^{-|x|/\nu_j^R} |x|^{- \ -1}, & x > 0, \\ 0, & x < 0. \end{cases} \qquad \mu^L(x) = \begin{cases} 0, & x > 0, \\ \lambda^L e^{-|x|/\nu_j^L} |x|^{- \ -1}, & x < 0. \end{cases} \tag{26}$$

so that the right-skewed jump component only allows up jumps and the left-skewed jump component only allows down jumps. We use different parameters $\lambda, \nu_j \in \mathbb{R}^+$ which is similar to CGMY model. Depending on the magnitude of the power coefficient α the sample paths of the jump process can exhibit finite activity ($\alpha < 0$), infinite activity with finite variation ($0 < \alpha < 1$), or infinite variation ($1 < \alpha < 2$). Therefore, this parsimonious specification can capture a wide range of jump behaviors. Further we put $\alpha = -1$, so the jump specification becomes a finite-activity compound Poisson process with an exponential jump size distribution as in Kou (1999).

Fair strike of the quadratic variation for three models

- For such Lévy density the characteristic exponent has the following form

$$
\begin{aligned}
{}^{R}_{x}(u) &= -iu\lambda^{R}\left[\left(\frac{1}{1-iu\nu_j^R} - \frac{\nu_j^R}{1-\nu_j^R}\right]\left(+ {}^{R}_{d}(u)\right.\right. \\
{}^{L}_{x}(u) &= iu\lambda^{L}\left[\left(\frac{1}{1+iu\nu_j^L} - \frac{\nu_j^L}{1+\nu_j^L}\right]\left(+ {}^{L}_{d}(u)\right.\right. \\
{}^{k}_{d}(u) &= \frac{1}{2}(\sigma^{k})^2(iu+u^2), \quad k = L, R,
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where ${}^{k}_{d}(u)$ is the characteristic exponent for the concavity adjusted diffusion component $\sigma W_t - \frac{1}{2}\sigma^2 t$.

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- Thus, form Eq. (27) we find that $({}^{k}_{x})''(0) \equiv (\sigma^k)^2 + 2\lambda^k n_j^k, \quad k = L, R.$

- Overall, the SSM model has 16 free parameters $\kappa^k, \theta^k, \eta^k, \rho^k, v_0^k, \sigma^k, \lambda^k, \nu_j^k, \quad k = L, R$ that can be obtained by calibrating the model to European option prices, again using the FFT method.

Fair strike of the quadratic variation for three models

- **NIG-CIR.** The normal inverse Gaussian distribution is a mixture of normal and inverse Gaussian distributions. The density of a random variable that follows a NIG distribution $X \approx NIG(\alpha, \beta, \mu, \delta)$ is given by (see Barndorf-Nielse (1998))

$$f_{NIG}(x; \alpha, \beta, \mu, \delta) = \frac{\delta \alpha e^{\delta \;+\; (x-\mu)}}{\pi \sqrt{\delta^2 + (x-\mu)^2}} K_1\left(\alpha \sqrt{\delta^2 + (x-\mu)^2}\right), \tag{28}$$

where $K_1(w)$ is the modified Bessel function of the third kind.

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- As a member of the family of generalized hyperbolic distribution, the NIG distribution is infinitely divisible and thus generates a Levy process $(L_t)_{t>0}$. For an increment of length s , the NIG Levy process satisfies

$$L_{t+s} - L_t \approx \mathrm{NIG}(\alpha, \beta, \mu s, \delta s) \tag{29}$$

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- Combined with the CIR clock change it produces a NIG-CIR model. The possible values of the parameters are $\alpha > 0, \delta > 0, \beta < |\alpha|$, while μ can be any real number. Below for convenience we use transformed variables, namely:

$$\Theta \equiv \beta/\delta, \quad \nu \equiv \delta\sqrt{\alpha^2 - \beta^2}$$

The characteristic exponent of the NIG model reads

$${}_x(u) = iu\mu + \delta\left[\sqrt{\alpha^2 - \beta^2} - \sqrt{\alpha^2 - (\beta + iu)^2}\right] \tag{30}$$

Results

- We use these three models to compute fair value of the quadratic variation contract on SMP500 index and Google on August 14, 2006. We calibrated them to the 480 available European option prices using differential evolution - a global optimization method. We found the following values of the calibrated parameters (see Tables (1-3)

κ	θ	η	v_0	ρ
1.572	0.038	0.504	0.019	-0.699

Table 1: Calibrated parameters of the Heston model

κ_L	θ_L	η_L	v_{0L}	ρ_L	σ_L	λ_L	ν_L
1.2916	0.6515	2.1152	0.3366	-0.9998	0.2077	0.02396	1.8455
κ_R	θ_R	η_R	v_{0R}	ρ_R	σ_R	λ_R	ν_R
6.7486	1.999	0.0004	0.0002	0.4049	0.0734	0.0029	0.5864

Table 2: Calibrated parameters of the SSM model

κ	θ	η	v_0	ρ	δ	ν	Θ	μ
2.855	0.093	0.787	0.057	-0.987	0.897	7.533	-1.285	0.482

Table 3: Calibrated parameters of the NIGCIR model

Results (continue)

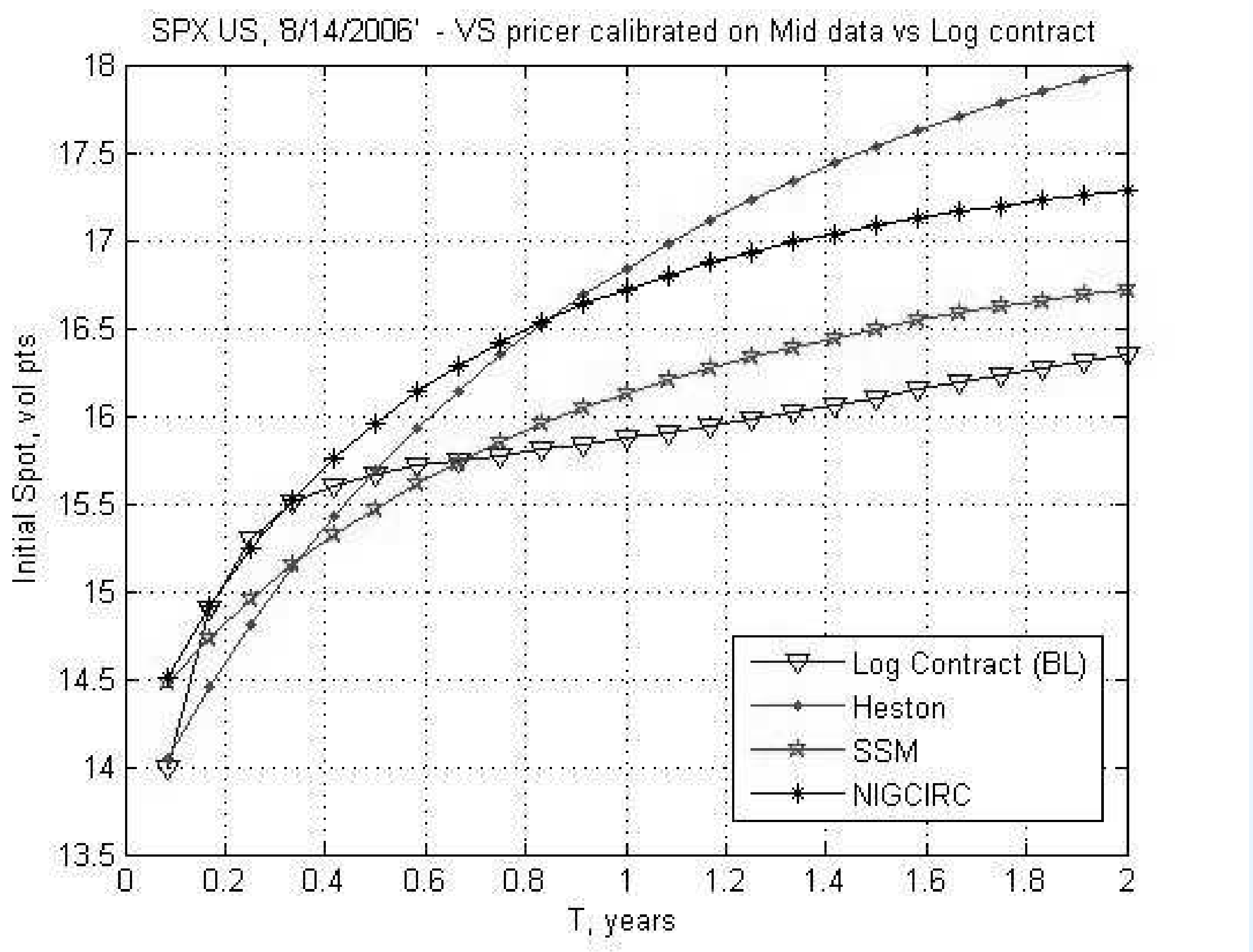

Figure 1: Fair strike of SPX in Heston, NIGCIR and SSM models. Comparison with a log contract (as per Bloomberg).

Results (continue)

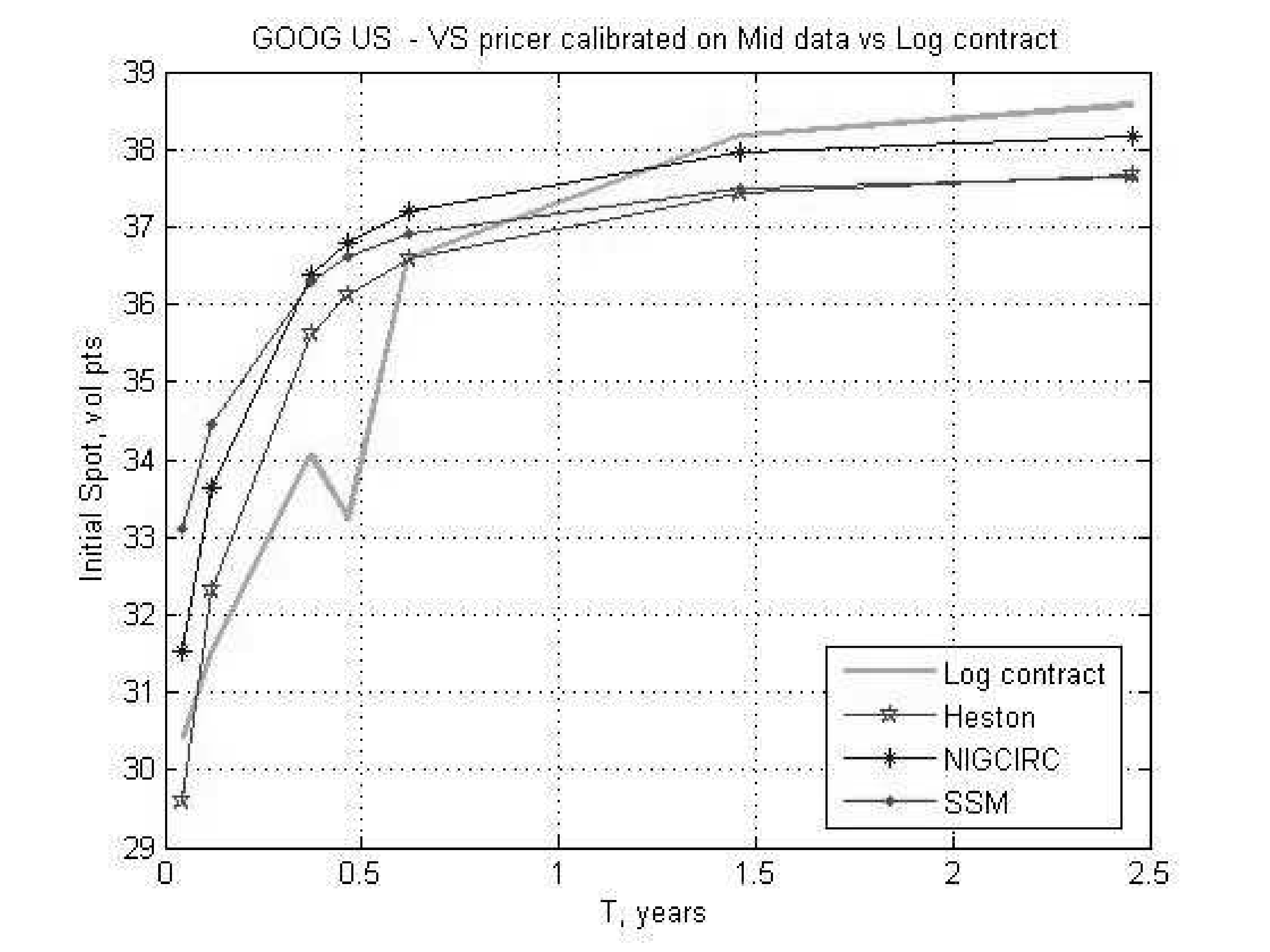

Figure 1: Same for Google

Volatility swaps

Volatility swaps

- Similar to a contract on the quadratic variation, a volatility swap contract makes a bet on the annualized realized volatility that is defined as follows

$$\overline{Vol}(s_t) \equiv \frac{1}{T}\mathbb{E}_{\mathbb{Q}}\sqrt{\sum_{i=1}^{N}\left[(s_{t_i} - s_{t_{i-1}})^2\right]} \approx \frac{1}{T}\mathbb{E}_{\mathbb{Q}}\left[\sqrt{\int_0^T \nu_t dt} \;\middle|\; \nu_0\right] = \frac{1}{T}\mathbb{E}_{\mathbb{Q}}[\sqrt{V}], \tag{31}$$

where V stays for the total annualized realized variance.

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  where V stays for the total annualized realized variance.
- Swishchuk (2004) uses the second order Taylor expansion for function $\sqrt{V}$ obtained in Brockhaus & Long (2000) to represent $\mathbb{E}_{\mathbb{Q}}[\sqrt{V}]$ via $\mathbb{E}_{\mathbb{Q}}[V]$ and $Var[V]$ as

$$\mathbb{E}_{\mathbb{Q}}[\sqrt{V}] \approx \sqrt{\mathbb{E}_{\mathbb{Q}}[V]} - \frac{VarV}{8(\mathbb{E}_{\mathbb{Q}}[V])^{3/2}}. \tag{32}$$

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- As we already showed for the CIR time-change the quadratic variation process V differ from that of the Heston model by the constant coefficient $(\ _x)''_u(0)$. Therefore, $Var[V]$ in our case differs from that for the Heston model by the coefficient $[(\ _x)''_u(0)]^2$. Thus, for the Lévy models with the CIR time-change the fair value of the annualized realized volatility is

$$\overline{Vol}(s_t) = \sqrt{(\ _x)''_u(0)}\ \overline{Vol}_H(s_t), \tag{33}$$

where $\overline{Vol}_H(s_t)$ is this value for the Heston model obtained by using the Eq. (32) and Eq. (20).

Volatility swaps (continue)

- A more rigorous approach is given by Jim Gatheral (2006). He uses the following exact representation

$$\mathbb{E}_{\mathbb{Q}}\left[\sqrt{V}\right] = \frac{1}{2\sqrt{\pi}} \int_0^\infty \frac{1 - \mathbb{E}_{\mathbb{Q}}\left[e^{-xV}\right]}{x^{3/2}} dx. \tag{34}$$

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- Here

$$\mathbb{E}_{\mathbb{Q}}\left[e^{-xV}\right] = \mathbb{E}_{\mathbb{Q}}\left[exp\left\{-x\int_0^T v_t dt\right\}\right]$$

is formally identical to the expression for the value of a bond in the CIR model if one substitutes there $\beta(\tau, \quad_x(u))$ with $-x$.

Options on the quadratic variation

Options on quadratic variation

- Having known the values of $\mathbb{E}_{\mathbb{Q}}[V]$ and $\mathbb{E}_{\mathbb{Q}}[\sqrt{V}]$ we can price vanilla European options on the quadratic variation using a log-normal method of Gatheral & Friz (2005). This method, however, first is an approximation, and second, for complicated models like SSM, accurate computing of $\mathbb{E}_{\mathbb{Q}}[\sqrt{V}]$ could be a problem.

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- We intend to proceed in sense of Roger Lee (2004) and make use of the FFT method. Let us denote

$$Q(T) \equiv \lambda \int_0^T \nu_t \, dt, \quad \lambda \equiv (\ \ _x)''_u(0)\frac{1}{T}. \tag{35}$$

For the CIR process the characteristic function $\phi(u,T) \equiv \mathbb{E}_{\mathbb{Q}}[e^{iuQ(T)}]$ is known

$$\begin{aligned} \phi(u,T) &= Ae^{B}, \quad B = \frac{2iu\lambda v_0}{\kappa + \delta \coth(\delta T/2)}, \\ A &= \exp\left[\frac{\kappa^2\theta T}{\eta^2}\right]\left[\cosh(\delta T/2) + \frac{\kappa}{\delta}\sinh(\delta T/2)\right]^{-\frac{2\kappa\theta}{\eta^2}}, \quad \delta^2 = \kappa^2 - 2iu\lambda\eta^2. \end{aligned} \tag{36}$$

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- Therefore, according to Lee (2004) the call option value on the quadratic variation is given by the following integral

$$C(K,T) = \frac{e^{-\;\log(K)}}{\pi}\int_0^\infty \mathrm{Re}\left[e^{-iv\log(K)}\omega(v)\right]dv, \quad \omega(v) = \frac{e^{-rT}\phi(v - i\alpha, T)}{(\alpha + iu)^2} \tag{37}$$

The integral in the first equation can be computed using FFT, and as a result we get call option prices for a variety of strikes.

Other affine activity rate models

Other affine activity rate models

- In a one factor setting, Carr and Wu adopt a generalized version of the affine term structure model proposed by Filipovic (2001), which allows a more flexible jump specification. The activity rate process ν_t is a Feller process with generator

$$\begin{aligned}\mathcal{A}f(x) &= \frac{1}{2}\sigma^2 x f''(x) + (a' - kx)f'(x) \\ &= \int_{\mathbb{R}_0^+} \left[f(x+y) - f(x) - f'(x)(1 \wedge y) \right] (m(dy) + x\mu(dy)),\end{aligned} \tag{38}$$

where $a' = a + \int_{\mathbb{R}_0^+}(1 \wedge y)m(dy)$ for some constant numbers $\sigma, a \in \mathbb{R}^+, k \in \mathbb{R}^+$ and nonnegative Borel measures $m(dy)$ and $\mu(dy)$ satisfying the following condition:

$$\int_{\mathbb{R}_0^+} (1 \wedge y)m(dy) + \int_{\mathbb{R}_0^+} (1 \wedge y^2)\mu(dy) < \infty. \tag{39}$$

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with the coefficients $\alpha(t,\ _x(u)), \beta(t,\ _x(u))$ given by the following ordinary differential equations:

$$\beta'(t) = \ _x(u) - k\beta(t) - \frac{1}{2}\sigma^2\beta^2(t) + \int_{\mathbb{R}_0^+}\left[1 - e^{-y\ \ (t)} - \beta(t)(1 \wedge y)\right]\mu(dy), \tag{41}$$
$$\alpha'(t) = a\beta(t) + \int_{\mathbb{R}_0^+}\left[1 - e^{-y\ \ (t)}\right]m(dy),$$

with boundary conditions $\beta(0) = \alpha(0) = 0$.

Other affine activity rate models

- **Theorem.** Given the above conditions the annualized quadratic variation of the Lévy process under stochastic time is

$$QV(s_t) = \frac{1}{T}\xi \mathbb{E}_{\mathbb{Q}}\left[\int_0^T \nu_t dt \,\Big|\, \nu_0\right] \equiv \frac{1}{T}\xi \mathbb{E}_{\mathbb{Q}}[V], \tag{42}$$

$$\xi \equiv (\ _x)''_u(0)\frac{\partial^2 \beta(t, \ _x(u))}{\partial t \partial u}(0,0) + (\ _x)'^2_u(0)\frac{\partial^3 \beta(t, \ _x(u))}{\partial t \partial^2 u}(0,0).$$

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Proof. We prove it based on the idea considered in the previous sections. Namely, we again express $QV(s_t)$ via the forward characteristic function $\phi_{t_{i-1},t_i}(u)$, which is

$$\phi_{t_{i-1},t_i}(u) = e^{iu(r-q)\tau}\mathbb{E}_{\mathbb{Q}}\left[\mathcal{L}^u_{\mathbf{T}_\tau}(\ _x(u)) \mid \nu_0\right] = \mathbb{E}_{\mathbb{Q}}\left[\exp\left[iu(r-q)\tau - \alpha(\tau, \ _x(u)) - \beta(\tau, \ _x(u))\nu_t\right] \mid \nu_0\right] \tag{43}$$

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Let us remind that $\kappa\tau \ll 1$ is a small parameter as well as $(r-q)\tau \ll 1$. Therefore we expand the above expression in series on τ up to the linear terms to obtain

$$\phi_{t_{i-1},t_i}(u) = \mathbb{E}_{\mathbb{Q}}\Big\{\exp\left[-\alpha(0, \;{}_x(u)) - \beta(0, \;{}_x(u))\nu_t\right]\cdot \tag{44}$$

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Note, that according to the boundary conditions $\alpha(0, \ _x(u)) = \beta(0, \ _x(u)) = 0$. Also from the second equation in the Eq. (41) we see that $\alpha'(\tau, \ _x(u))_\tau(0, \ _x(u)) = 0$. Substituting these terms into the Eq. (44) and computing $-\partial^2\phi_{t_{i-1},t_i}(u)/\partial^2 u(u=0)$ we find

$$\frac{\partial^2 \phi_{t_{i-1},t_i}(u)}{\partial u^2}\Big|_{u=0} = \mathbb{E}_{\mathbb{Q}}[\xi\tau\nu_t]. \tag{45}$$

■

3/2 power clock change

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- In this section we consider one more class of the stochastic clock change. Despite it is not affine, it still allows variance swaps to be priced in a closed form. Originally this model has been proposed in a simple form (long term run coefficient is constant) by Heston Lewis (2000) to investigate stochastic volatility. Here we consider a more general case when the long-term run could be either a deterministic function of time, or even a stochastic process.

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- Let the futures price F of the underlying asset be a positive continuous process. By the martingale representation theorem, there exists a process v such that:

$$\frac{dF_t}{F_t} = \sqrt{v_t} d\tilde{Z}_t, \qquad t \in [0, T], \tag{46}$$

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- In particular, let us assume the risk-neutral process for instantaneous variance to be:

$$dv_t = \kappa v_t(\theta_t - v_t)dt + \epsilon v_t^{3/2} d\tilde{W}_t, \qquad t \in [0, T], \tag{47}$$

where $\tilde{W}$ is a $\mathbb{Q}$ standard Brownian motion, whose increments have known constant correlation $\rho \in [-1, 1]$ with increments in the $\mathbb{Q}$ standard Brownian motion $\tilde{Z}_t$, i.e.

$$d\tilde{Z}_t d\tilde{W}_t = \rho dt, \qquad t \in [0, T]. \tag{48}$$

3/2 clock change (continue)

- The 3/2 power specification for the volatility of v is empirically supported. The v process is mean-reverting with speed of mean reversion κv_t, where κ is known. The reason that the speed of mean-reversion is proportional rather than constant is primarily for tractability

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- Although our primary motivation for proportional speed of mean-reversion is tractability, nonlinear drift in the v process is also empirically supported.
- In the CIR model the stochastic time change is linear in drift. Therefore, the variance swap fair value is independent of how the volatility of V_t is specified. In contrast, when the drift of V_t is nonlinear, e.g. quadratic then the answer depends on how the volatility of V_t is specified.

CF

- We show that the conditional Laplace transform of the risk-neutral density of the realized quadratic variation is given by:

$$C^L(\lambda, I_t) \equiv L(t, v) = \frac{\Gamma(\gamma - \alpha)}{\Gamma(\gamma)} \left(\frac{2}{\epsilon^2 I_t v}\right) M\left(\alpha; \gamma; \frac{-2}{\epsilon^2 I_t v}\right), \tag{49}$$

where

$$\alpha \equiv -\left(\frac{1}{2} + \frac{\kappa}{\epsilon^2}\right) + \sqrt{\left(\frac{1}{2} + \frac{\kappa}{\epsilon^2}\right)^2 + 2\frac{\lambda}{\epsilon^2}},$$

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- To derive a closed-form solution for the variance swap price under the stochastic 3/2-power clock change. we again follow our method. Now $\theta = \theta(t)$ is a known deterministic function of time. Again we consider the forward characteristic function of an arbitrary Levy process with the characteristic exponent $\;_x(u)$ under the stochastic clock change determined by the "3/2-power" law. Similarly to Eq. (44)

$$\phi_{t_{i-1}, t_i}(u) = e^{iu(r-q)\tau} \mathbb{E}_{\mathbb{Q}}\left[\mathcal{L}^u_{\mathbf{T}_\tau}(\;_x(u)) \,\middle|\, \nu_0\right] = \mathbb{E}_{\mathbb{Q}}\left[e^{iu(r-q)\tau} C^L(\;_x(u), I_{t_i}) \,\middle|\, \nu_0\right] \tag{51}$$

3/2 model (continue)

- Now we make an assumption that $\kappa\theta(t)\tau \ll 1$ is a small parameter. This is a generalization of the assumption $\kappa\tau \ll 1$, that we made for the CIR clock change, for the case of the "3/2- power" model. Therefore, we expand the above expression in series on $\kappa\theta(t)\tau$ up to the linear terms.

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- First of all, expansion of $I_{t_{i-1}}$ reads

$$I_{t_i} = \tau + \kappa\theta(\tau)\tau^2 + O(\tau^2), \tag{52}$$

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- As per Abramowitz & Stegun (1964) (13.5.1) an asymptotic expansion series for $M(\alpha;\gamma;z)$ at large $|z|$ reads

$$M(\alpha;\gamma;z) = \frac{e^{i\pi}\ \Gamma(b)}{\Gamma(b-a)} z^{-} \left[\sum_{n=0}^{R-1} \left(\frac{(\alpha)_n (1+\alpha-\gamma)_n}{n!} (-z)^{-n} + O\left(|z|^{-R}\right)\right]\right) + \frac{e^z \Gamma(b)}{\Gamma(a)} z^{-} \left[\sum_{n=0}^{S-1} \left(\frac{(\gamma-\alpha)_n (1-\alpha)_n}{n!} (-z)^{-n} + O\left(|z|^{-S}\right)\right]\right), \quad -\frac{3}{2}\pi < argz < \frac{3}{2}\pi. \tag{54}$$

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We keep the first two terms in these series with $n = 0, 1$ Further, omitting a tedious algebra and remembering that $\Gamma(0) = \infty$ we find that

$$-\frac{\partial^2 \phi_{t_{i-1},t_i}(u)}{\partial u^2}\bigg|_{u=0} = (\quad x)''_u(0)\mathbb{E}_{\mathbb{Q}}[\tau v_{t_i}]. \tag{55}$$

3/2 model (continue)

- Using this formula together with the Eq. (6) we obtain exactly the same result as for the CIR process, i.e.

$$QV(s_t) = (\quad _x)''_u(0)\frac{1}{T}\sum_{i=1}^{N}\mathbb{E}_{\mathbb{Q}}\left[\tau\nu_{i-1}|\nu_0\right] \approx (\quad _x)''_u(0)\mathbb{E}_{\mathbb{Q}}\left[\frac{1}{T}\int_0^T \nu_t dt \mid \nu_0\right] \equiv (\quad _x)''_u(0)\mathbb{E}_{\mathbb{Q}}[V]. \tag{56}$$

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- The only difference is that now $\mathbb{E}_{\mathbb{Q}}[V]$ is computed using the "3/2-power" law, rather than the CIR process. This can be done by using the conditional Laplace transform of realized variance which for this process is available in the closed form (Itkin & Carr, 2007). Indeed, we have

$$L(t,v) \equiv \mathbb{E}_{\mathbb{Q}}\left[e^{-\lambda\int_t^T v_u du}\middle| v_t = v\right], \qquad v \geq 0, t \in [0,T]$$

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$$\begin{aligned}\mathbb{E}_{\mathbb{Q}}[V] &\equiv \mathbb{E}_{\mathbb{Q}}\left[\int_t^T v_u du \,\Big|\, v_t = v_0\right] = -\frac{\partial L(t,v)}{\partial \lambda}\Big|_{\lambda=0} \qquad (57)\\ &= \left[-\log\left(\frac{2}{\epsilon^2 I_T v_0}\right) + \frac{\Gamma'(2\nu)}{\Gamma(2\nu)} - 2M^{(0,1,0)}\left(0;2\nu;-\frac{2}{\epsilon^2 I_T v_0}\right) - M^{(1,0,0)}\left(0;2\nu;-\frac{2}{\epsilon^2 I_T v_0}\right)\right]\frac{\partial\alpha}{\partial\lambda}\Big|_{\lambda=}\\ &= \left[-\log\left(\frac{2}{\epsilon^2 I_T v_0}\right) + \frac{\Gamma'(2\nu)}{\Gamma(2\nu)} - M^{(1,0,0)}\left(0;2\nu;-\frac{2}{\epsilon^2 I_T v_0}\right)\right]\frac{2}{2\kappa+\epsilon^2},\end{aligned}$$

3/2 model (continue)

- Using this formula together with the Eq. (6) we obtain exactly the same result as for the CIR process, i.e.

$$QV(s_t) = (\ x)''_u(0)\frac{1}{T}\sum_{i=1}^{N}\mathbb{E}_{\mathbb{Q}}\left[\tau\nu_{i-1}\,|\nu_0\right] \approx (\ x)''_u(0)\mathbb{E}_{\mathbb{Q}}\left[\frac{1}{T}\int_0^T \nu_t\,dt \mid \nu_0\right] \equiv (\ x)''_u(0)\mathbb{E}_{\mathbb{Q}}[V]. \tag{56}$$

- The only difference is that now $\mathbb{E}_{\mathbb{Q}}[V]$ is computed using the "3/2-power" law, rather than the CIR process. This can be done by using the conditional Laplace transform of realized variance which for this process is available in the closed form (Itkin & Carr, 2007). Indeed, we have

$$L(t,v) \equiv \mathbb{E}_{\mathbb{Q}}\left[e^{-\lambda\int_t^T v_u\,du}\middle|v_t = v\right], \qquad v \geq 0,\ t \in [0,T]$$

and thus

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}}[V] &\equiv \mathbb{E}_{\mathbb{Q}}\left[\int_t^T v_u\,du\,\middle|\,v_t = v_0\right] = -\frac{\partial L(t,v)}{\partial\lambda}\Big|_{\lambda=0} \qquad (57)\\
&= \left[-\log\left(\frac{2}{\epsilon^2 I_T v_0}\right) + \frac{\Gamma'(2\nu)}{\Gamma(2\nu)} - 2M^{(0,1,0)}\left(0;2\nu;-\frac{2}{\epsilon^2 I_T v_0}\right) - M^{(1,0,0)}\left(0;2\nu;-\frac{2}{\epsilon^2 I_T v_0}\right)\right]\frac{\partial\alpha}{\partial\lambda}\Big|_{\lambda=0}\\
&= \left[-\log\left(\frac{2}{\epsilon^2 I_T v_0}\right) + \frac{\Gamma'(2\nu)}{\Gamma(2\nu)} - M^{(1,0,0)}\left(0;2\nu;-\frac{2}{\epsilon^2 I_T v_0}\right)\right]\frac{2}{2\kappa+\epsilon^2},
\end{aligned}$$

where

$$\nu = 1 + \frac{\kappa}{\epsilon^2}, \qquad I_T \equiv \int_0^T e^{\kappa\int_0^{t'}\theta(u)du}\,dt',$$

$M^{(1,0,0)}(\alpha,\gamma,\zeta)$ is the derivative of $M(\alpha,\gamma,\zeta)$ on α, $M^{(0,1,0)}(\alpha,\gamma,\zeta)$ is the derivative of $M(\alpha,\gamma,\zeta)$ on γ, $\Gamma'(2\nu) \equiv d\Gamma(x)/dx|_{x=2\nu}$, and $M^{(0,1,0)}(0,\gamma,\zeta) = 0$.

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- We considered several examples which includes the Heston model, SSM, NIGCIR for the CIR clock change, and 3/2-power clock change.

Thank you!